Class forcing and topos theory[1]

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 The video of the talk on which these notes are based is available at `www.youtube.com/watch?v=4AaSySq8-GQ`

It is well-known that forcing over a model of material set theory corresponds to taking sheaves over a small site (a poset, a complete Boolean algebra, and so on). One phenomenon that occurs is that given a small site, all new subsets created are smaller than a fixed bound depending on the size of the site. There is a more general notion of forcing invented by Easton to create new subsets of arbitrarily large sets, namely class forcing, where one starts with a partially ordered class. The existing theory of class forcing is entirely classical, with no corresponding intuitionist theory as in ordinary forcing. Our understanding of its relation to topos theory is in its infancy, **[[** but it is clear that class forcing is about taking small sheaves on a large site, or rather, considering colimits of large diagrams of sheaf toposes and their inverse image functors. **]]**[2] That these do not automatically form a topos means that the theory has interesting twists and turns. This talk will outline the theory of class forcing from a category/topos point of view, give examples and constructions, and finally a list of open questions – not least being whether an intuitionistic version of Easton's theorem on the continuum function holds.

[2] **Added April 2017:** this is incorrect! Jensen forcing gives new sets(=sheaves) which aren't set-generic(=small). My thanks to Joel David Hamkins for patiently explaining this to me.

Terminology and disclaimers

Disclaimers In what follows, we need to refer to categories of various sizes. To forestall any size issues we can assume two Grothendieck universes, $\mathbb{U} \in \mathbb{V}$. Elements of $\mathbb{U}$ will be *small*, elements of $\mathbb{V}$ will be *moderate*. Anything not an element of $\mathbb{U}$ will be said to be *large*, and anything not an element of $\mathbb{V}$ will be *very large*. This is more of a convenience than a strictly necessary device, as the potentially very large categories we will come across could be treated as being models of the first-order theory of elementary toposes instead.

A second point is that all elementary toposes in these notes will be (small) cocomplete, relative to a base topos of sets, and so have a natural number object.

Terminology In dealing with set theory through the lens of topos theory, it is convenient to have words to distinguish the different sorts of sets or set theories. The following do not have formal definitions, but we can give examples of each.

1. **Material** sets — these are sets where elements have independent existence from sets that contain them. Primordial examples are ZF(C), BZ(C). The latter set theory is Bounded Zermelo (with Choice), which is well-known to be equiconsistent with ETCS (see below).

2. **Structural** sets — these are sets where everything is invariant under isomorphism, and where elements do not have independent existence. Sometimes set theories of this sort have been referred to as *categorical* set theories, but this is not strictly necessary. Examples include Lawvere's ETCS (elementary theory of the category of sets) and Shulman's SEAR (sets, elements and relations). The latter is not axiomatised as a category with certain properties, but as a three-sorted theory using the eponymous sorts.[3] Structural set theories can be augmented with axiom schemas that make them equiconsistent with ZFC, if desired.

[3] See the nLab page `http://ncatlab.org/nlab/show/SEAR` for details of SEAR.

The continuum function from an ahistorical viewpoint

In this section we consider purely classical logic and assume sets satisfy the axiom of choice, as is usually considered in traditional set theory. Take material or structural sets as desired.

If we let Set_{inj} to be the category of sets and injections, then we denote by $\mathrm{Card}_{\leq}$ to be the partial order reflection. This latter partial order is the category of *cardinals*. There is a canonical functor

$$\mathrm{Set}_{inj} \longrightarrow \mathrm{Card}_{\leq}$$

sending a set to its cardinal. Using well-orderings of sets we can get a section of this functor, sending a cardinal to the least ordinal in bijection with it,[4] but this is not particularly needed for what follows. Notice that the covariant powerset functor $P\colon \mathrm{Set} \to \mathrm{Set}$ (sending a function f to $\exists_f$) restricts to a functor $P\colon \mathrm{Set}_{inj} \to \mathrm{Set}_{inj}$; by the universal property of partial order reflection one gets a commutative square

[4] If we do not have the axiom of choice, but still have a Von Neumann-style cumulative hierarchy, we can use the so-called *Scott's trick* to identify a set representing each cardinality.

$$\begin{tikzcd}
\mathrm{Set}_{inj} \arrow[r] \arrow[d, "P"'] & \mathrm{Card}_{\leq} \arrow[d, "2^{(-)}"] \\
\mathrm{Set}_{inj} \arrow[r] & \mathrm{Card}_{\leq}
\end{tikzcd}$$

The functor here denoted $2^{(-)}$ is the *continuum function*.[5] Cantor's continuum hypothesis asks about the behaviour of the continuum function at $\mathbb{N}$, considered as a cardinal:

[5] In a more general setting, one might write $\Omega^{(-)}$ instead.

$$\exists ? A \text{ such that } \mathbb{N} < A < 2^{\mathbb{N}}$$

Famously, Gödel produced a model of ZFC (the constructible universe L) in which for any cardinality κ, any A such that $\kappa \leq A \leq 2^\kappa$ must be either κ or 2^κ (so the Generalised Continuum Hypothesis is true in that model). Likewise, Cohen gave a model of ZFC in which there *is* an A such that $\mathbb{N} < A < 2^{\mathbb{N}}$, and his method, *forcing*, can be used to construct variants along these lines, subject to an important limitation to be described below.

More generally, one can wonder what the global behaviour of the continuum function is: does it skip values? which ones can it skip? which values can it take? There are two cases:

- *Regular cardinals*: this case has been 'solved', by Easton, in the sense that we know all possible behaviours of the continuum function on the regular cardinals
- *Singular cardinals*: it is a hard and open problem to fully determine the possible behaviour of the continuum function on singular cardinals. The area of pcf theory deals with this part, and we won't say more about this here.[6]

[6] See eg Menachem Kojman, *PCF Theory*, Topology Atlas Invited Contributions vol. 6 issue 1 (2001) p. 74–77, `http://at.yorku.ca/t/a/i/c/44.htm`.

The power of power objects

But first, why should we care about the continuum function as topos theorists? The reason is that almost all of the properties of toposes (can be seen to) arise from the existence of *power objects*.

Definition 1. An *elementary topos* is a category with

1. finite limits
2. power objects: for all objects X there is an object PX and a relation $\in_X \hookrightarrow X \times PX$ such that for any other relation $R \hookrightarrow X \times Y$ there is a unique morphism $\rho\colon Y \to PX$ and a pullback square

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \in_X \arrow[d] \\ X \times Y \arrow[r, "id\times\rho"'] & X \times PX \end{tikzcd}$$

Power objects can be[7] assumed or chosen to be (contravariantly) functorial.

[7] up to the usual caveats

Very little logical structure is afforded by finite limits, but we know the internal logic of an elementary topos is full higher order intuitionistic logic, and much of mathematics can be described therein. Thus the continuum function, the partial order reflection of the power object functor, captures a lot of information about what objects of toposes are and how they behave.

For the purposes of the rest of these notes, it will be helpful to give another definition of elementary topos that highlights a different breakdown of the topos properties, but again singling out power objects.

Definition 2. A category E is an *infinitary Heyting pretopos with subobject classifier* if:

- it has finite limits
- it has (small) disjoint sums stable under pullback
- it is a Heyting category[8]: the internal logic is first-order intuitionistic
- it has a subobject classifier Ω (in the usual sense)

The first two properties make E (infinitary) *lextensive*. 'Small sums' is not an elementary property, and can only be defined in the presence of a fixed base topos.

[8] amongst other things, the subobject lattices are Heyting algebras, and one has the usual quantifiers/connectives available; see `https://ncatlab.org/nlab/show/Heyting+category`.

Given two infinitary Heyting pretoposes E, F as above, a morphism[9] from E to F is a lex, cocontinuous functor $F \to E$.

[9] an *algebraic* morphism, as emphasised by Joyal at the conference: the 'inverse image' part of an imaginary geometric morphism.

We say an infinitary Heyting pretopos E has *finitary W-types* if it has initial algebras for finitary polynomial endofunctors of the form

$$X \mapsto \sum_{n \in \mathbb{N}} A_n \times X^n.$$

Finitary W-types are also initial algebras for the polynomial endofunctors defined using the dependent product $\Pi_{\mathbf{n}}$, arising from cardinals $\mathbf{n}\colon A \to B$ in the slice topos over B.

We say it has *parameterised* finitary W-types if it has finitary W-types and these remain initial algebras (on pulling back along $Y \to 1$) for the induced polynomial endofunctors on E/Y.

Example 1. Examples of finitary W-types are as follows:

1. natural numbers objects
2. free monoids
3. free monoid actions
4. list objects

Such a pretopos is good for studying strongly predicative mathematics. It is closely related to the unpublished notion of *arithmetic universe* due to Joyal, and that of *list-arithmetic pretopos* of Maietti et al. One can think of it as being quite like a presentable category, minus the actual presentability (i.e. generation under colimits of a set of compact objects)

Note that there are no internal homs in general, so there are no power objects, even though there is a subobject classifier. If we add exponentials $\Omega^{(-)}$, then E becomes a (cocomplete) elementary topos.

Forcing in a nutshell

To be blunt, the various types of forcing can *all* be performed using sheaves. The general pattern of ordinary forcing (again, ahistorically) is as follows:

This was pointed out by Ščedrov in his 1984 AMS Memoir *Forcing and Classifying Topoi*, but was no doubt known earlier.

$$\begin{array}{ccccc} \mathbb{P} & \xrightarrow{\text{take sheaves}} & \mathrm{Sh}(\mathbb{P},\neg\neg) & \xrightarrow{\mathrm{Sub}(1)\xrightarrow{\Phi}\mathbf{2}} & \mathrm{Sh}(\mathbb{P},\neg\neg)_{\Phi} \\ \text{small poset} & & \text{using double negation topology} & & \text{filterquotient construction} \\ & & | & & | \\ & & \text{(work in internal logic)} & & \text{(topos is now well-pointed)} \\ & & \downarrow & & \downarrow \\ & & \text{Boolean-valued model} & & \text{forced model of BZC/ZFC} \end{array}$$

Note that the apparent simplicity of the above recipe hides the sophisticated infinitary combinatorics that can go into the construction of $\mathbb{P}$ and proving properties of the resulting model.

The vertical arrows are given by a delicate and technical construction due to in various measures to Cole, Osius, Fourman, Hayashi and Mitchell, and give back a material set theory.

There are other types of forcing, given by other topos-theoretic constructions:

- Equivariant sheaves for a continuous action of a topological group $G \to \mathrm{Aut}(\mathbb{P})$: the resulting model of material set theory is called a *symmetric submodel*.
- The topos $\mathrm{Cont}(G)$ continuous actions of a topological group G on sets. The resulting model of set theory with atoms[10] is called a *permutation model*.
- Taking sheaves on a complete Boolean algebra gives something equivalent to forcing over a poset, but this raises the possibility of taking sheaves over a complete Heyting algebra instead: this gives *Heyting valued models* of intuitionistic material set theory.

[10] Such 'forcing' models are due to Fraenkel and Mostowski and deal with the theory ZFCA: there sets can have elements called *atoms* that have no elements themselves, but are not empty.

The key result to take from this section is that forcing like this can only affect *set-many*[11] values of the continuum function. **But:** Easton's solution of the continuum problem[12] used a *large*, or class-sized, partial order $\mathbb{P}$. This means he could adjust class-many values of $2^{(-)}$, essentially one at a time. This technique is now known as *class forcing*. As any category theorist knows, taking sheaves on a large site, as it appears we would need to do, moves to a larger universe due to size problems.

[11] In the sense there is a family of sets, parameterised by a set, at which the values of $2^{(-)}$ are shifted up or down as desired, with little or no control on all other values. One can compute an upper bound on this cardinality in terms of the cardinality of the site at hand.

[12] W.B. Easton, *Powers of regular cardinals*, Annals of Mathematical Logic **1** (1970) 139–178.

Class forcing using fibred sites

This is not a disaster, though. Compare the 'faux topos' of SGA4.1,[13] which is built using a progroup $(G_i | i \in I)$ with I large (or a proper class), and is essentially the topos of continuous actions on sets of the large topological group "$\lim_{i\in I} G_i$"[14].

[13] SGA 4, IV.2.8 – this is an example of what we now call a non-Grothendieck topos with a geometric morphism to Set.

[14] For an example in practice, see the author's paper *The Weak Choice Principle WISC may Fail in the Category of Sets*, Studia Logica **103** (2015) 1005–1017.

To get around the issue of considering a large site of definition, consider the following

Setup: large filtered category $\mathcal{R}$ with initial object, and a diagram of sites:

$E\colon \mathcal{R}^{\mathrm{op}} \longrightarrow \mathrm{Site_{fib}} \xrightarrow{\mathrm{Sh}} \mathrm{Topos_b}$	$\mathrm{Site_{fib}}$ morphisms: $\quad (D,K) \underset{T}{\overset{F}{\rightleftarrows}} (C,J)$, T is cover-preserving

The morphisms of $\mathrm{Site_{fib}}$ are *fibrations of sites*, introduced by Moerdijk[15] to treat 2-categorical limits of toposes. The simplest example, namely a projection $\mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ of posets with top element, equipped with the double-negation topology, and its right adjoint, turns out to encompass many examples in classical set theory. Every geometric morphism between Grothendieck toposes arises from a fibration of sites, as long as one is happy to change the site of definition for the domain topos.

[15] I. Moerdijk, *Continuous fibrations and inverse limits of toposes*, Composito Mathematica **58** (1986) 45–72.

Remark. Taking the perspective that a choice of site for a Grothendieck topos is like choosing a basis for a vector space, fibrations of sites can be considered as a kind of 'normal form' for geometric morphisms.

The diagram of sites is what is called a *fibred site* in SGA 4, and the resulting diagram of toposes is called a *fibred topos*[16]. When the diagram is small (i.e. $\mathcal{R}$ is small) then one can consider the 'total topos', which is the 2-categorical limit of the diagram of toposes.

[16] See eg SGA 4.1 VI.7.1, where the treatment is in terms of fibred categories rather than indexed categories. With the usual caveats, these two approaches are equivalent. Note that Verdier in his Lecture Notes in Mathematics uses the notion described here.

Ščedrov, in his thesis[17] showed that *set*-iterated *set*-forcing is, on

[17] Published as: A. Ščedrov, *Forcing and classifying topoi*, Memoirs of the AMS no. 295 (1984).

the structural side, working with **fibred toposes**. Iterated forcing in the traditional sense is, put simply, a process of doing one forcing (hence sheaves on one site), then for a poset in the new model (so an internal site in the topos) performing another forcing, and then for another poset in the *new* model, forcing again, and so on. This can be iterated either finitely or infinitely many times. Using such techniques one can (for example) iteratively adjust values of the continuum function, ascending up a given family of (regular) cardinals. Other kinds of infinitely iterated forcing can be used to eliminate infinite combinatorial structures that are otherwise not destroyed by single instances of forcing.

But if $\mathcal{R}$ is large? What do we do? Consider the case that $\mathcal{R}$ is well-founded for simplicity. Form the diagram

$\mathrm{LEX}^{\mathrm{cocont}}$ is the very large 2-category of large infinitary lextensive categories with functors preserving finite limits and colimits. It contains locally presentable categories, which include Grothendieck toposes.

$$E^* \colon \mathcal{R} \longrightarrow \mathrm{Topos}_{\mathrm{b}}^{\mathrm{op}} \hookrightarrow \mathrm{LEX}^{\mathrm{cocont}}$$

where the inclusion of toposes and (bounded) geometric morphisms into lex categories and lex, cocontinuous functors takes the inverse image functor. We want to form $\operatorname{colim}_{\mathcal{R}} E^*$:

$$\begin{tikzcd}
a \arrow[d] & \mapsto & E(a) \arrow[r, shift right] \arrow[d, shift right] & \operatorname{colim}_{\mathcal{R}} E^* =: \mathcal{E} \arrow[l, dashed, shift right] \arrow[dl, dashed, shift left] \\
b & \mapsto & E(b) \arrow[u, shift right, "f^*"'] \arrow[ur, shift left] &
\end{tikzcd}$$

There may be no dashed functors as shown, which would be the right adjoints: the lex category $\mathcal{E}$ is not guaranteed to be locally small. There are, however, nontrivial examples that give rise to locally small $\mathcal{E}$, and such that the resulting direct image functor gives a geometric morphism that is not bounded.

The lack of local smallness means we can't apply the adjoint functor theorem as in the case for Grothendieck toposes.

Theorem 1. Let $E\colon \mathcal{R}^{\mathrm{op}} \to \mathrm{Topos}_{\mathrm{b}}$ be a filtered diagram with $\mathcal{R}$ (classically) well founded with **initial object**. Then $\mathcal{E} = \operatorname{colim}_{\mathcal{R}} E^*$ is an infinitary Heyting pretopos with subobject classifier and parameterised finitary W-types.

Let $E\colon \mathcal{R}^{\mathrm{op}} \to \mathrm{Topos}_{\mathrm{b}}$ be a filtered diagram of toposes. For $d \in \mathcal{D}$ and an object $X \in E(d)$, say that ⌜powersets of X are eventually constant⌝ if there is $j\colon d \to d'$ such that for all $k\colon d' \to d''$, the canonical comparison map

$$k^* P(j^* X) \to P(k^* j^* X))$$

is an isomorphism, where we have written $k^* := E(k)^*$ etc for short. Further, say ⌜powersets are eventually constant⌝ if ⌜powersets of X are eventually constant⌝ for every X.

Theorem 2. Given a filtered diagram $E\colon \mathcal{R}^{\mathrm{op}} \to \mathrm{Topos_b}$ of toposes, if ⌜powersets are eventually constant⌝, then $\mathcal{E}$ is a cocomplete elementary topos.

This seems like a lot of conditions to check, but the following lemma shows we can cut things down a fair bit

Lemma 1. Let $f\colon E \to F$ be a geometric morphism, and X an object of F such that the canonical map $f^*(PX) \to P(f^*X)$ is an isomorphism. Then for any subquotient Y of X, the canonical $f^*PY \to P(f^*Y)$ is also an isomorphism.

In the classical set case, sites are posets together with the double-negation topology. The resulting categories of sheaves are localic (so every object is a subquotient of a constant sheaf), which is an important special case where we can say something stronger.

Corollary 1. If $E(a)$ is localic for all $a \in \mathcal{R}$, and ⌜powersets of κ are eventually constant⌝ for all κ in the base topos[18], then $\mathcal{E}$ is a cocomplete elementary topos.

[18] In fact, one only needs this to hold at a a cofinal sequence of such κ, so for example, at a cofinal sequence of regular cardinals.

Definition 3. A fibration of sites $F\colon D \rightleftarrows C : T$ is (κ, B)-*Easton*, for $\kappa \in \mathrm{Set}$, $B \in \mathrm{Sh}(C)$, if the canonical map

$$F^*(P(\kappa \times B)) \to P(F^*(\kappa \times B))$$

is an isomorphism, where we have omitted the 'constant sheaves' functor applied to κ. In the localic case, we write κ-Easton for $(\kappa, 1)$-Easton.

In actual fact, this is a definition of a property of a geometric morphism, but we are mostly interested when a geometric morphism arises from a given fibration of sites.

Example 2. Fix κ regular and let $\mathbb{P}$ be a poset with top such that every $\neg\neg$-sieve contains a covering sieve generated by $\leq \kappa$ elements. Let $\mathbb{Q}$ be a poset with top such that every $\lambda^{\mathrm{op}} \to \mathbb{Q}$ bounded below for all $\lambda \leq \kappa$. Then $\mathbb{P} \times \mathbb{Q} \to \mathbb{P}$ is κ-Easton.

Example 3. A poset map $\mathbb{P}' \to \mathbb{P}$ is κ-Easton if $(\mathbb{P}', \neg\neg)$ **in the internal logic** of $\mathrm{Sh}(\mathbb{P}, \neg\neg)$, has $\bigcap_{i\in\kappa} S_i$ a covering sieve, where each S_i is a covering sieve.

Since for B a *bound* for a topos $p\colon E \to \mathrm{Set}$, every object is a subquotient of $p^*\kappa \times B$, for some $\kappa \in \mathrm{Set}$, one can make a stronger statement than Theorem 2.

Note that if λ is a subquotient of κ in Set then (κ, B)-Easton implies (λ, B)-Easton for any B.

Theorem 3. Given a filtered diagram of sites $S\colon \mathcal{R}^{\mathrm{op}} \to \mathrm{Site}_{\mathrm{fib}}$, then if for all $a \in \mathcal{R}$ in some cofinal class and every κ in some subquotient-cofinal class in Set, every $S(b) \to S(a)$ is (κ, B)-Easton, for B some bound for $\mathrm{Sh}(S(a))$, then ⌜powersets are eventually constant⌝, and the resulting pretopos as in Theorem 1 is a cocomplete elementary topos.

Again, in the localic case, things simplify, as each B_a can be taken as terminal for localic toposes.

Future directions

Moerdijk used his notion of fibration of sites to give several simple characterisations of properties of geometric morphisms at the site level. We seek such a characterisation for the property of being (κ, B)-Easton.

Question 1. Can we find a site-level characterisation of what it means to be (κ, B)-Easton, in the sense of conditions on the fibration of sites $F\colon D \rightleftarrows C : T$, rather than on the geometric morphism or sheaves?

Question 2. Can we generalise $\kappa \times B$ in the definition of (κ, B)-Easton fibration to a more general family (i.e. not just a product)? Or is this even something that we need to do? Compare to embedding and representation theorem for bounded toposes.

Recall that in practice, the B in $\kappa \times B$ is a *bound* for the induced geometric morphism between toposes.

In the literature, the most easily understood class forcings are *sequential* class forcings, but there are other more mysterious class forcings that are not of this form. Almost all attention is on **pretame class forcings**,[19] and then **tame** class forcings are those that result in powersets existing in the class-forced model (cf the topos-theoretic theorems above). We give an informal definition of a pretame class forcing for the sake of completeness:

[19] See e.g. Section 2 of Sy D. Friedman, *Constructibility and Class Forcing*, in: Handbook of Set theory, Eds. Matthew Foreman and Akihiro Kanamori, Springer 2010. Available from `http://www.logic.univie.ac.at/~sdf/papers/class-forcing.pdf`.

Definition 4. Let $\mathbb{P}$ be a large partial order. We say $\mathbb{P}$ is *pretame* if for every set-indexed family $(D_i)_{i\in I}$ of subclasses $D_i \subset \mathbb{P}$ whose down-closures are dense below some $p \in \mathbb{P}$, there is a $q \leq p$ and a family $(d_i)_{i\in I}$ of sub*sets* $d_i \subset D_i$ such that the down-closure of each d_i is dense below q.

It is the author's view that being able to approach arbitrary pretame forcings using the technology of fibred sites and toposes would give a conceptual clarity.

Conjecture. Every pretame class forcing partial order can be written as the colimit of the right adjoint functors in a large filtered diagram $\mathcal{R}^{\mathrm{op}} \to \mathrm{Site}_{\mathrm{fib}}$ of (fibrations of) sites such that $\mathcal{R}$ has all initial segments small, and conversely, every such large filtered diagram of (fibrations of) sites gives rise to a pretame class forcing.

Does a pretame class forcing have no *ORD*-chains?

[[**Added April 2017:** Note that it is *not* the case that every tame class-forced model arises as in Theorem 1, since Jensen Coding[20] must involve non-small sheaves for its class forcing partial order, and it is tame, hence pretame.]]

[20] See eg S.D. Friedman, *A simpler proof of Jensen's coding theorem*, Annals of Pure and Applied Logic **79** (1994) 1–16.

Question 3. BONUS QUESTION: does every geometric morphism arise from a suitable choice of large fibred topos?

Given that Easton's theorem only applies to sets in classical logic, one can ask whether, for sets in non-classical logic (for instance, IZF or CZF), there is an analogue of Easton's theorem. Note that if one merely drops the axiom of choice (so working in ZF), there are results analogous to Easton's using variants[21] of the continuum function and on embedding partially-ordered classes (with a mild 'niceness' condition) into the class of cardinals, ordered either by injections or surjections[22]. Note that the first of these results only applies to *initial ordinals*, which excludes non-well-orderable sets. Karagila's result uses the choiceless definition of cardinals using *Scott's trick*,[23] but says nothing about the continuum function. It is unclear whether there is a meaningful version of Easton's theorem working with arbitrary constructive sets, or in an arbitrary topos, but one might seek out such a result.

[21] A. Fernengel and P. Koepke, *An Easton-like theorem for Zermelo-Fraenkel Set Theory without Choice*, arXiv:1607.00205.

[22] A. Karagila, *Embedding orders into the cardinals with* DC_κ, Fundamenta Mathematicae, **226** (2014) 143–156.

[23] This technique chooses a canonical set representing each bijection class, using the cumulative hierarchy and notion of rank of a set.